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LOCALIZATION LIMITERS IN TRANSIENT PROBLEMS

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Abstract—In materials with strain softening, closed form solutions have shown that the energy dissipation associated with the localization and damage processes vanishes. For finite element solutions, the dissipation depends on the size of the element in the localized zone. To remedy this situation, non-local and imbricate formulations have previously been proposed. In this paper, a simple remedy based on introducing additional higher order terms in the governing equations in the strain-softening portions of the domain is proposed. It is shown that the resulting formulation provides solutions which are independent of element size with a finite energy dissipation. Example solutions are given for one-dimensional rod problems and one-dimensional spherically symmetric problems.

1. INTRODUCTION

When deformed far enough into the inelastic range, certain materials exhibit narrow zones of intense straining. This localization of deformation may be linked in many cases, in particular for one-dimensional models, to a strain-softening effect in the constitutive behavior of the material, that is a negative slope in the stress–strain curve.

As will be seen subsequently, the appearance of localization in classical local continuum mechanics with rate independence is associated, from a mathematical point of view, with a change of type of the governing equations: loss of ellipticity in quasi-static problems; change from hyperbolic to elliptic type in the dynamic case. This change of type allows the prediction of the critical stress level which triggers localization; unfortunately, it leaves the size of the localization zone unspecified in static problems and gives infinite strains over a set of measure zero in dynamic problems[1, 2]. When incorporated into a computational model, strain-softening behavior therefore leads to severely mesh-dependent results, in which deformation localizes in one element irrespective of its size. Furthermore, the energy dissipated in the strain-softening domain tends to zero as the mesh is refined.

Several methods have been proposed to remedy this undesirable situation. In Ref. [3], a non-local formulation was used based on an averaging procedure, and for one-dimensional dynamic problems, a localization limited to a domain of finite size, and an energy dissipation that remains constant with mesh refinement were achieved. This averaging approach has also been used recently in Ref. [4], where the averaged quantities are the damage parameters in damage-type constitutive laws. Another procedure of a similar type consists of adding gradients of state variables: in Ref. [5], the gradient of the yield function was included in the constitutive equation. Triantafyllidis and Aifantis[6] used a second deformation gradient-dependent term in the expression of the strain–energy function to restore ellipticity, and obtained analytic solutions in the particular case of the Blatz–Ko material. Alternatively, Needleman illustrated in a recent paper[7] how, for a simple one-dimensional problem, the introduction of material rate dependence in the constitutive model via viscoplasticity can also result in momentum equations which remain elliptic and hyperbolic in the static and dynamic cases, respectively.

These methods, as well as others that have been proposed in recent years, ultimately try to eliminate mesh sensitivity in numerical calculations by ensuring that the size of the localization zone remains finite, and can therefore be referred to as localization limiters. In this paper, a localization limiter is proposed which is based on the introduction of an additional higher order term in the governing equations in the strain-softening portions of the domain. Attention is focused on models for which localization is directly related to the onset of strain softening, namely dynamic one-dimensional problems.

The paper is organized as follows: Section 2 briefly reviews the mathematical aspects and difficulties underlying the localization phenomenon. Section 3 introduces the localization limiter based on the addition of high order terms to the strain expression. Examples are presented in Section 4, where the localization limiter is used in one-dimensional rod problems and spherically symmetric converging wave problems. This last type of problem is considered again in Section 5, which analyzes solutions obtained by using a material rate-dependent approach.

2. LOCALIZATION AND CHANGE OF TYPE IN THE GOVERNING EQUATIONS

From a mathematical point of view, the localization phenomenon is associated with a change of type of the governing partial differential equations. This change of type can be of various nature.

(1) In static problems, the onset of localization is connected with the loss of ellipticity in the equilibrium equations[8, 9]. In this approach, localization is viewed as an instability process: critical conditions are determined, at which the constitutive relations allow a bifurcation from a homogeneous state of deformation into a shear band with high strain values. This bifurcation is related to the loss of ellipticity of the incremental equilibrium equations. It should be pointed out that in multidimensional problems, the onset of localization is not necessarily linked with strain softening (a decline in the stress value with increasing strain), but indeed localization can occur with a positive slope in the stress–strain relation, in the presence of certain geometric or constitutive factors, such as non-associative flow rules[10, 11].

(2) In the wave propagation context, the emergence of localization is linked with a change of type in the governing wave equation, this time from hyperbolic to elliptic. Let us consider for example the one-dimensional wave equation

$$\sigma(\varepsilon)_{,x} = \rho u_{,tt}, \quad 0 \leqslant x \leqslant L \tag{1}$$

where ρ is the material mass density, u the displacement, σ the stress and ε the strain, related to the displacement through the compatibility condition

$$\varepsilon = u_{,x}. \tag{2}$$

Equations (1) and (2) can be written as a system of first-order differential equations

$$\begin{bmatrix} v \\ \varepsilon \end{bmatrix}_{,t} + \begin{bmatrix} 0 & -\dfrac{\sigma(\varepsilon)}{\rho} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ \varepsilon \end{bmatrix}_{,x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

where the velocity is

$$v = u_{,t}. \tag{4}$$

If the function σ' is strictly positive, eqn (3) is a strictly hyperbolic system, and in the particular case of a linear elastic material (where $\sigma'(\varepsilon) = E_t$, the tangent Young's modulus), we find the usual expression for the wave speed c_0

$$c_0 = \sqrt{\left(\frac{E_t}{\rho}\right)}. \tag{5}$$

If, however, the tangent modulus $\sigma'(\varepsilon)$ decreases to zero for a certain strain ε_p, system (3) loses strict hyperbolicity and, further, if $\sigma'(\varepsilon) < 0$, it becomes elliptic. The corresponding initial value problem, now elliptic, is considered to be ill posed because the wave speed is imaginary. From a physical point of view, this change of type results in an accumulation of strain into a band or "deformation trapping", as described by Wu and Freund[12].

In both cases, static and dynamic, this change of type in the governing equations raises mathematical difficulties as well as numerical ones: when finite element solutions are computed, some of the results appear to be strongly dependent on the mesh refinement, since there is no finite length scale associated with the localization. Furthermore, the energy dissipation vanishes as the size of the element goes to zero, and in closed form solutions[1, 2], it is shown that the dissipation vanishes because it occurs over a set of measure zero. It is of interest to note that the imaginary character of the wave speed is not a critical shortcoming since in closed form solutions[1, 2] the strain softening is always limited to a set of measure zero.

(3) The difficulties due to a change of type in the governing equations are not confined to the area of solid mechanics. In fluid dynamics, classical transonic problems are governed by the transonic small disturbance equation[13]

$$[1 - M_\infty^2 - (\gamma+1)M_\infty \phi_{,x}]\phi_{,xx} + \phi_{,yy} = 0 \tag{6a}$$

that can be rewritten as

$$A\phi_{,xx} + \phi_{,yy} = 0 \tag{6b}$$

where A is the non-linear coefficient appearing in eqn (6a), M_∞ the free stream Mach number, ϕ the velocity potential, and γ the ratio of specific heat coefficients; eqn (6b) is an equation of mixed type, it changes from hyperbolic to elliptic when A changes from negative (supersonic flow) to positive (subsonic flow). This change of type also causes numerical difficulties and is often treated by a regularization term consisting of higher order derivatives.

3. A LOCALIZATION LIMITER BASED ON HIGHER ORDER TERMS

As it has been already pointed out, the energy dissipation vanishes for dynamic solutions involving strain softening because the localization takes place over a set of measure zero. To remedy this situation, localization limiters have been introduced. A localization limiter should ensure that energy dissipation remains finite at arbitrary mesh refinement so that damage localization to a region of zero volume does not take place. One way this can be achieved is by inserting additional gradient terms into the momentum equation. More precisely, we will consider for the sake of clarity the one-dimensional equation of motion

$$\sigma_{,x} = \rho u_{,tt}. \tag{7}$$

We replace the usual strain–displacement relation

$$\varepsilon = u_{,x} \tag{8}$$

by the following expression involving a higher order term:

$$\hat{\varepsilon} = \varepsilon + \alpha\varepsilon_{,xx} \tag{9a}$$

or

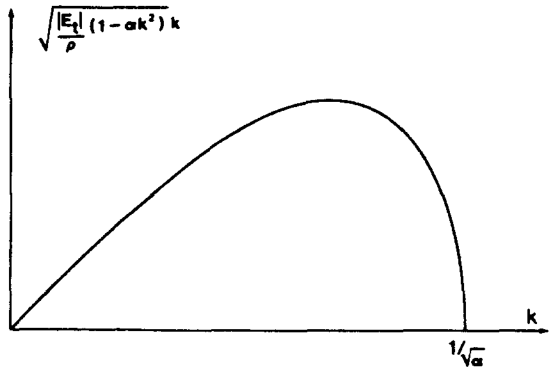

Fig. 1. Linearized analysis: plot of $\gamma(k) = \{(|E_t|/\rho)(1-\alpha k^2)\}^{1/2}k$ as a function of wavelength k.

$$\bar{\varepsilon} = u_{,x} + \alpha u_{,xxx} \tag{9b}$$

where α is a constant which will determine the extent of localization control.

We consider a linearized analysis of a disturbance δu, in which the material is in a strain-softening state over an interval $[x_1, x_2]$. Then

$$\delta\sigma(x) = -|E_t|\,\delta\varepsilon(x) \quad \text{for } x \text{ in } [x_1, x_2] \tag{10}$$

where the tangent modulus $E_t < 0$ is assumed constant. We will call the second term in the modified strain expression (9a) the *localization limiter*.

The equation of motion (7) becomes, for a small disturbance δu, using eqns (9b) and (10)

$$\delta u_{,tt} + \frac{|E_t|}{\rho}(\delta u_{,xx} + \alpha\delta u_{,xxxx}) = 0. \tag{11}$$

If we look for a possible wave solution of this last equation (using a von Neumann procedure[14])

$$\delta u(x,t) = A\,\mathrm{e}^{ik(x-vt)} \tag{12}$$

where k is the wave number and v the phase velocity, we obtain

$$k^2v^2 = -\frac{|E_t|}{\rho}(1-\alpha k^2)k^2 \tag{13}$$

or

$$kv = \mathrm{i}\left\{\frac{|E_t|}{\rho}(1-\alpha k^2)\right\}^{1/2} k. \tag{14}$$

We can plot (Fig. 1) $\gamma(k) = \{(|E_t|/\rho)(1-\alpha k^2)\}^{1/2}k$ as a function of the wave number k. For high frequencies, that is a wave number $k \geqslant 1/\sqrt{\alpha}$, $\gamma(k)$ is imaginary; therefore kv is real and the disturbance δu is bounded. For low frequencies ($k < 1/\sqrt{\alpha}$), kv is purely imaginary, and therefore δu grows unboundedly.

However, the region of the material where strain softening occurs is presumably small compared to a characteristic length of the problem (the total length of the rod); in fact in certain closed form solutions it is confined to a single point as $\alpha \to 0$. Therefore, large wavelengths will not be able to develop in the strain-softening region, and the above analysis shows that the higher order strain will bound any growth of short wavelength inputs. This very simplified analysis illustrates the effectiveness of the additional higher order term in limiting the localization.

This effect can also be understood more qualitatively: near a peak in the local maximum in the strain function, the second derivative will be negative, therefore its presence will tend to reduce the strain or, equivalently, to increase the stress, since we are in the strain-softening branch of the stress–strain curve.

A simplified perturbation analysis sheds light on the relationship between the parameter α and the width δ of the localization zone. The modified equation of motion

$$u_{,tt}+\frac{|E_t|}{\rho}(u_{,xx}+\alpha u_{,xxxx})=0 \tag{15}$$

can be nondimensionalized by introducing $x^*=x/L$

$$\frac{\partial^2 u}{\partial t^2}+\frac{|E_t|}{\rho L^2}\left(\frac{\partial^2 u}{\partial x^{*2}}+\mu\frac{\partial^4 u}{\partial x^{*4}}\right)=0 \tag{16}$$

where $\mu=\alpha/L^2$.

The localization limiter appears as a singular perturbation term, since the coefficient of the highest order derivative in the equation is much smaller than the other coefficients ($\mu \ll 1$). Hence, the effect of the additional term is negligible everywhere, except over a length δ where the second- and fourth-order terms should balance each other. We define a new scaling

$$\xi=\frac{x^*}{\mu^a}. \tag{17}$$

The balancing is given by

$$\frac{\partial^2 u}{\partial x^{*2}}\sim\mu\frac{\partial^4 u}{\partial x^{*4}} \tag{18a}$$

or

$$\mu^{-2a}u_{,\xi\xi}\sim\mu^{-4a+1}u_{,\xi\xi\xi\xi}. \tag{18b}$$

As μ tends to zero, the spatial derivative terms balance asymptotically when

$$a=\tfrac{1}{2}. \tag{19}$$

The size δ of the localization domain is therefore proportional to $\sqrt{\alpha}$, i.e.

$$\delta\sim p\alpha^{1/2},\quad p=\text{constant}. \tag{20}$$

It should be pointed out that the localization limiter proposed in eqn (9b) is related to the Lax–Wendroff method[15] used in fluid mechanics in order to trigger shocks within a narrow confine of the finite difference grids. In fact, eqn (15) is a mathematical counterpart of the transient advection equation with artificial diffusion

$$\phi_t+u\phi_x+\alpha\phi_{xx}=0 \tag{21}$$

analyzed by Roache[16].

In another domain of application, higher order derivatives have also been used in the past for the backwards heat equation, known to be ill posed. In the context of phase transformations in metal alloys (the "spinodal decomposition" problem), Cahn and Hilliard[17] defined a generalized diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) - 2K\frac{\partial^4 u}{\partial x^4}. \tag{22}$$

The additional fourth-order term in eqn (22), which accounts for the increase of energy due to the gradient of the concentration u, prevents the appearance of instability in the solution when the diffusion coefficient $D(u)$ becomes negative. Here again, a higher order term is introduced to deal with a change of type in the governing equations.

The localization limiter defined in eqn (9a) is related to the strain averaging technique used in the non-local theory through a Taylor expansion[18]. In the one-dimensional non-local theory, the non-local or average strain $\bar{\varepsilon}$ at a point x is defined as

$$\bar{\varepsilon}(x) = \frac{1}{l}\int_{-l/2}^{l/2} \varepsilon(x+s)\,\mathrm{d}s. \tag{23}$$

We can write an expansion around the center value $\varepsilon(x)$

$$\varepsilon(x+s) = \varepsilon(x) + s\varepsilon_{,x}(x) + \frac{s^2}{2}\varepsilon_{,xx}(x) + o(s^2) \tag{24a}$$

which, inserted in eqn (23), yields

$$\bar{\varepsilon}(x) = \varepsilon(x) + \frac{l^2}{24}\varepsilon_{,xx}(x) + o(l^2). \tag{24b}$$

The localization limiter $\bar{\varepsilon}$ appears as the lowest order differential approximation of the limiter based on a strain averaging procedure over a length l, with

$$l = \{24\alpha\}^{1/2} \tag{25}$$

in the one-dimensional case. It is of interest to compare expressions (20) and (25); evidently the domain of averaging l is related to the size of the domain of localization.

4. NUMERICAL IMPLEMENTATION AND EXAMPLES

The initial value problem, eqn (3) was solved by the finite element method. We used the usual weak (variational) form of eqn (1) (all natural boundary conditions are considered homogeneous)

$$\int_0^L [\delta u_{,x}\sigma + \delta u \rho u_{,tt}]\,\mathrm{d}x = 0. \tag{26}$$

This displacement field was then approximated by

$$u(x,t) = N_I(x)u_I(t) \tag{27}$$

where u_I are the nodal displacements, and repeated upper case indices are summed over the nodes. This yields the discrete equations

$$N_{I,x}\sigma\,\mathrm{d}x+\int_0^L \rho N_I N_J u_{J,tt}\,\mathrm{d}x=0. \tag{28}$$

This can be written as

$$M_{IJ}u_{J,tt}+f_I^{\mathrm{int}}=0 \tag{29a}$$

where

$$f_I^{\mathrm{int}}=\int_0^L N_{I,x}\sigma\,\mathrm{d}x \tag{29b}$$

$$M_{IJ}=\int_0^L \rho N_I N_J\,\mathrm{d}x. \tag{29c}$$

A tangent stiffness is not used in the numerical implementation since an explicit time integrator is used, but it is interesting to write it down since it indicates that the system is not self-adjoint. The tangent stiffness is defined by

$$f_{I,t}^{\mathrm{int}}=K_{IJ}^{\tan}u_{J,t} \tag{30}$$

so using eqns (29b), (27) and (9) we obtain

$$K_{IJ}^{\tan}=\int_0^L N_{I,x}E_t(N_{J,x}+N_{J,xxx})\,\mathrm{d}x \tag{31a}$$

where E_t is the tangent modulus. This tangent stiffness matrix, that would appear in an implicit treatment of the transient problem, could be transformed by an integration by parts into a self-adjoint operator and an additional boundary term

$$K_{IJ}^{\tan}=\int_0^L N_{I,x}E_tN_{J,x}\,\mathrm{d}x-\int_0^L N_{I,xx}E_tN_{J,xx}\,\mathrm{d}x+[N_{I,x}E_tN_{J,xx}]_0^L. \tag{31b}$$

The introduction of the second-order derivative in the strain expression, and therefore of a fourth-order displacement derivative in the equation of motion would require in theory the use of C^1 (Hermite) shape functions. As we wish only to illustrate the effectiveness of the method, we have used simple linear displacement (C^0) finite elements. The second derivative is computed by a centered finite difference approximation

$$(\varepsilon_{,xx})_j=(\varepsilon_{j+1}-2\varepsilon_j+\varepsilon_{j-1})/\Delta x^2 \tag{32}$$

and we have used uniform (constant Δx) meshes. Calculations were conducted with an explicit time integrator, namely a central difference method with a lumped mass matrix. We used for most calculations a Courant number $CFL = 0.7$–0.9, although, in certain cases, a lower value 0.4 had to be used to obtain stable results.

An interesting aspect of the method is that the localization limiter is introduced only in the region where strain softening occurs: this is implemented by initializing at zero an array of values $\alpha(j)$, $j = 1, NELE$ where $NELE$ is the number of elements, and attributing the value $\alpha(j) = \alpha$ only when element j has reached a strain-softening state. Therefore, in the regions of the body where strain softening is not reached, the strain considered in the equation of motion is the usual local strain ε, without any higher order derivatives.

Two additional features were incorporated in the method. It should first be noted that

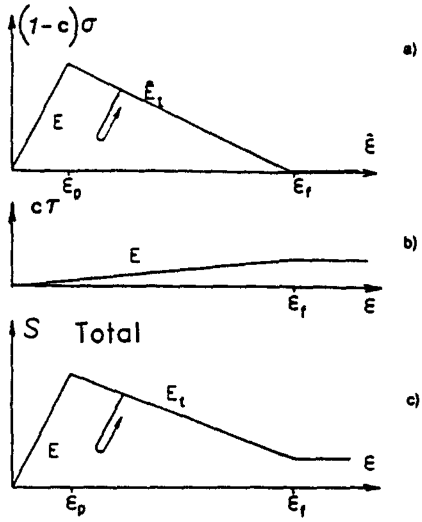

Fig. 2. Combination of stress–strain laws. For the one-dimensional rod problem, E is Young's modulus; for the spherically symmetric problem, E is replaced by K (bulk modulus), E_T by K_T, ε by ε_v (volumetric strain).

$$\varepsilon + \alpha\varepsilon_{,xx} = 0 \tag{33}$$

admits as a solution

$$\varepsilon_1(x) = A \cos \frac{1}{\sqrt{\alpha}} x + B \frac{1}{\sqrt{\alpha}} x \tag{34}$$

where A and B can be arbitrarily large. This solution ε_1, called a zero-energy mode, can appear in the numerical solution; when superposed on the "real" solution, it leads to meaningless results. The same problem arises in the finite element implementation of non-local methods based on strain averaging[3], and has been resolved so far only by combining a local and a non-local law. Similarly, in our case, we will combine two stress–strain relations

$$\sigma = \hat{C}(\hat{\varepsilon})\hat{\varepsilon} \tag{35a}$$

$$\tau = C(\varepsilon)\varepsilon \tag{35b}$$

where σ is the stress resulting from the localization limiter $\hat{\varepsilon}$, τ the stress resulting from the usual strain ε, $\hat{C}(\hat{\varepsilon})$ and $C(\varepsilon)$ account respectively for a constitutive relation with strain softening and one without it, as summarized in Fig. 2.

The stress used in the equation of motion is taken as a combination

$$S = (1-c)\sigma + c\tau. \tag{36}$$

The term $c\tau$ acts as a stabilizing factor, preventing the development of zero energy modes. Parameter c is typically of the order of 0.1.

Second, in order to obtain solutions in the presence of the higher order derivatives, it is necessary to add damping. When wave propagation phenomena are analyzed by finite elements, oscillations from one element to the next appear ahead of and behind of the wave front, in both time and space (Gibbs' oscillations)[19]. These oscillations would lead to erratic second derivatives, and we therefore prevent them by introducing damping via

$$f_I^{\text{int}} = \int N_{I,x} S \, dx + \beta \int N_{I,x} \dot{S} \, dx \tag{37}$$

where parameter β controls the amount of damping introduced. The above represents a stiffness proportional damping, which acts on the high frequencies introduced by the Gibbs phenomenon, whereas mass proportional damping would affect low frequencies.

It should be pointed out that the *raison d'être* of these two additional features is purely related to numerical aspects, and that they are not intrinsic to the method itself. As far as damping is concerned for example, Sandler and Wright have shown[20] that in certain situations (wave propagation in a rod) damping alone can act as a localization limiter with a certain effectiveness. Reference [21] showed however the limitations of that approach, and its inappropriateness in other situations such as in the spherically symmetric problem described in Section 4.2.

The localization limiter introduced here can handle these two types of problems successfully, proving therefore that it is not the damping term alone that accounts for the localization limiting effect.

4.1. *Wave propagation in a rod*

This problem was considered in Ref. [3] (Fig. 3(a)). Equal and opposite velocities v_0 are applied to the two ends of a rod of length $2L$ made of a strain-softening material, so that tensile waves are generated at the ends. The magnitude of the strain is slightly less than the strain corresponding to the onset of strain softening. These tensile waves propagate elastically to the center; when they meet at the center, the stress would double if the behavior remained elastic, so that strain softening starts at this midpoint.

The analytical solution for this problem was proposed in Ref. [1]: localization occurs at the midpoint where the strain becomes infinite. The solution, symmetric about the midpoint $x = L$, is expressed for the left half as

$$u = -v_0\left\langle t - \frac{x}{c_0}\right\rangle - v_0\left\langle t - \frac{2L-x}{c_0}\right\rangle \tag{38a}$$

$$\varepsilon = \frac{v_0}{c_0}\left[H\left(t - \frac{x}{c_0}\right) - H\left(t - \frac{2L-x}{c_0}\right) + 4\langle c_0 t - L\rangle \delta(x-L)\right] \tag{38b}$$

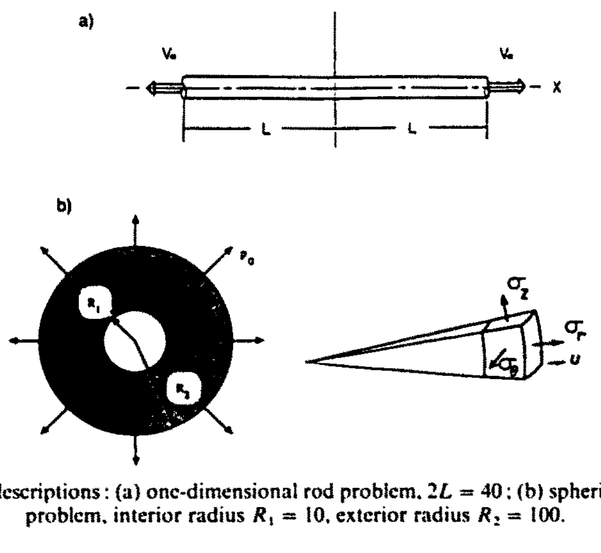

Fig. 3. Problem descriptions: (a) one-dimensional rod problem, $2L = 40$; (b) spherically symmetric problem, interior radius $R_1 = 10$, exterior radius $R_2 = 100$.

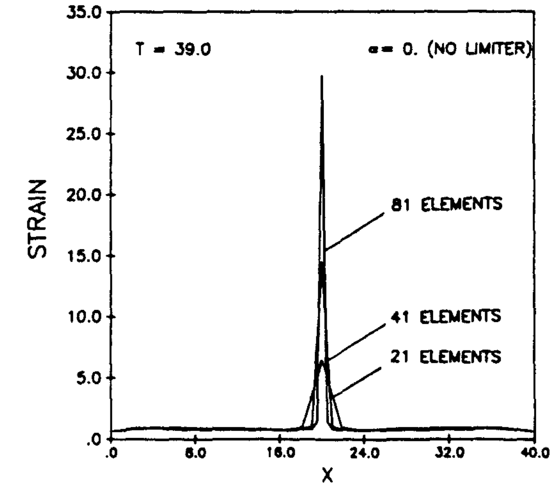

Fig. 4. One-dimensional rod problem, strain plots at time $T = 39.0$ for different meshes, no higher order term limiter ($\alpha = 0$).

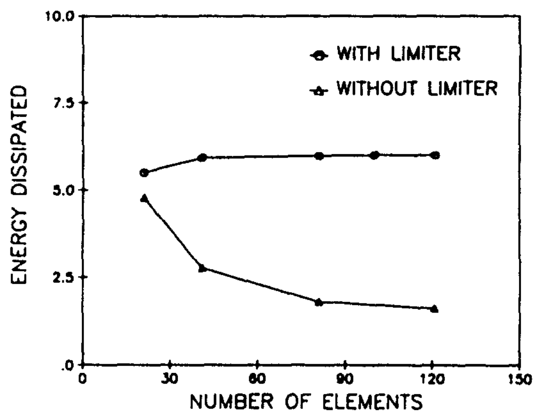

Fig. 5(a). One-dimensional rod problem, energy dissipated in the rod vs number of elements, with and without localization limiter.

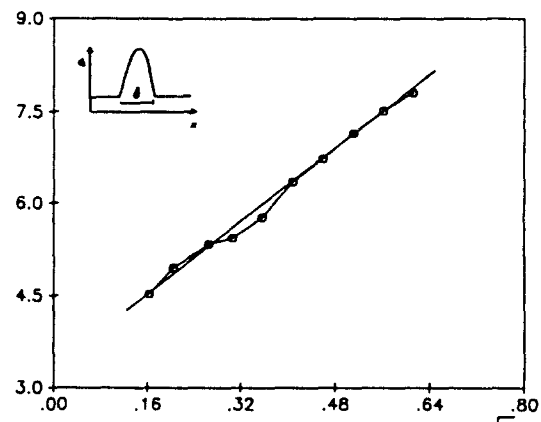

Fig. 5(b). One-dimensional rod problem, size δ of the localization zone as a function of $\sqrt{\alpha}$.

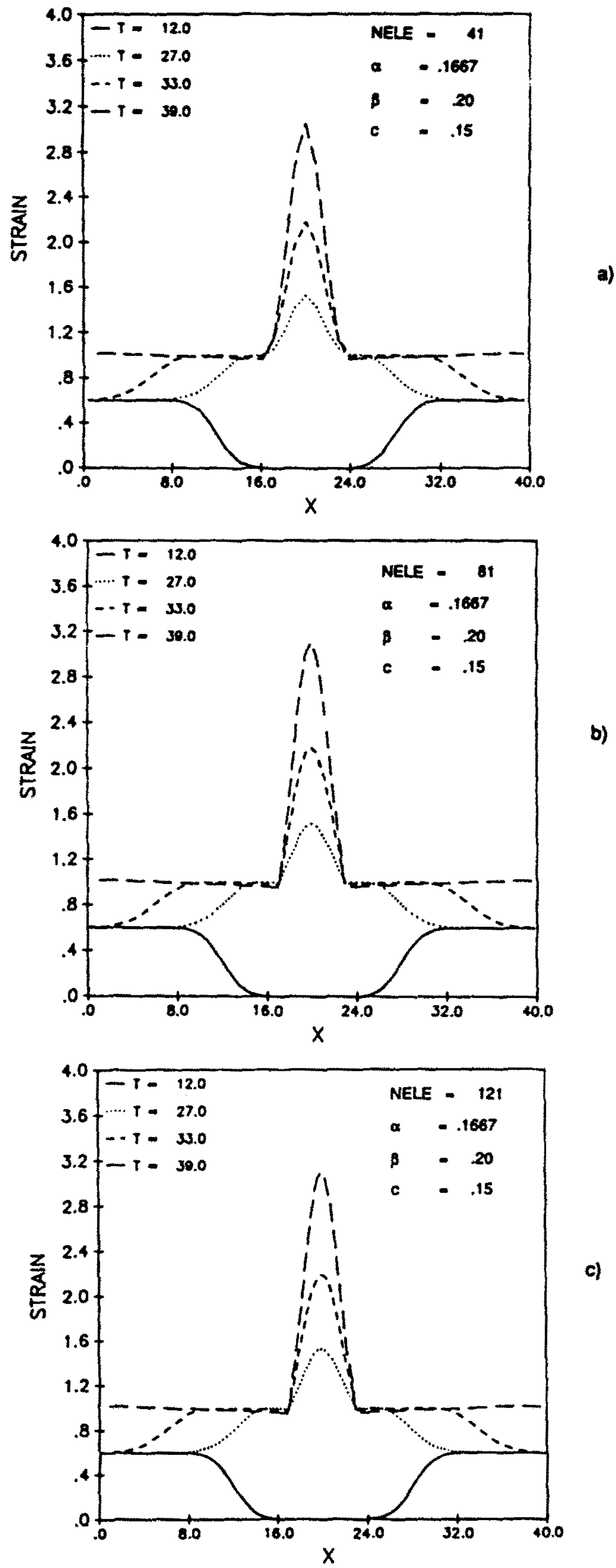

Fig. 6. One-dimensional rod problem, strain plots obtained using the localization limiter ($\alpha = 0.1667$): (a) 41 elements; (b) 81 elements; (c) 121 elements. For all plots, combination factor $c = 0.15$, damping coefficient $\beta = 0.20$.

where H is the Heaviside step function, $\langle A \rangle = A$ if $A > 0$, $\langle A \rangle = 0$ otherwise, δ is the Dirac-delta function, c_0 the elastic wave speed in the material. Numerical studies of this problem based on non-local approaches were conducted in Refs [3, 4]. Here we will use the localization limiter

$$\hat{\varepsilon} = \varepsilon + \alpha \varepsilon_{,xx}. \tag{9a}$$

The combination of stress–strain laws considered to stabilize the zero energy modes is illustrated in Fig. 2, and we used a damping factor $\beta = 0.20$, and a combination parameter $c = 0.15$, in accordance with the results of Ref. [3]. Other parameters used in the calculations were: density $\rho = 1$, end velocity $v_0 = 0.6$, and for the stress–strain relations in Fig. 3, $E = 1$, yield stress $\sigma_p = 1$, $E_t = -0.25$, $\varepsilon_f = 5$, nearly horizontal tail of slope $E_f = 0.001$ beyond ε_f.

It was first checked (Fig. 4) that, without introducing the localization limiter (that is, for $\alpha = 0$), the strain profiles are severely dependent on the mesh refinement, and the localization zone shrinks to one element, irrespective of its size. Furthermore, the total energy dissipated in the mesh tends to zero as the mesh is refined, as seen in Fig. 5(a). Convergence studies were then performed with the localization limiter, for a value $\alpha = 0.1667$, for different meshes with an increasing number of elements (Figs. 6(a)–(c)). They exhibit a localization limited to a finite zone, the length of that zone and the strain profiles being independent of the mesh refinement. Moreover, the total energy dissipated in the rod, computed as in Ref. [3] by

$$W(t_{r+1}) = W(t_r) + \sum_{j=1}^{NELE} \frac{h_j}{2} (1-c)[\sigma_{j,r}\varepsilon_{j,r} - \sigma_{j,r+1}\varepsilon_{j,r+1} + (\sigma_{j,r} + \sigma_{j,r+1})(\varepsilon_{j,r+1} - \varepsilon_{j,r})] \tag{39}$$

is independent of the mesh size, all other parameters remaining equal (in eqn (39), r refers to the time step, and h_j the size of element j). This is illustrated in Fig. 5(a).

Calculations were also conducted at fixed mesh size for different values of α (Fig. 5(b)), showing that the length of the localization zone is *linearly dependent on* $\sqrt{\alpha}$ (the line does not extrapolate to the origin (0, 0) since the presence of the damping coefficient β partially limits the localization[20], even for $\alpha = 0$). This is consistent with the results of Refs [3, 4], which found a linear dependence in l (averaging length), since the Taylor expansion (23) yielded a linear relation between α and l^2.

4.2. *Spherically symmetric problem*

This problem (Fig. 3(b)) was considered with strain-softening materials in Ref. [21]. A sphere made of a strain-softening material is loaded with a uniform traction on its exterior surface. To better appreciate the complexity of this problem, consider the load to be a ramp function in time. Before the onset of strain softening at an interior surface S, a portion of the stress will have passed through S. Due to the spherical geometry, the stresses in this wave are amplified as they pass to the center and trigger the formation of additional strain-softening surfaces. As conjectured in Ref. [21], it seems that an infinite number of localization surfaces will appear, although no analytical solution has been proposed so far.

The localization limiter defined previously in eqn (9a) was used to solve numerically this problem. We considered a sudden application of a uniform normal traction $\sigma_r = p_0 H(t)$ at the exterior surface, where H is the Heaviside step function. The applied surface pressure was chosen as $p_0 = 0.708$; for this boundary condition, the wave propagating from the outer surface remains elastic until the wavefront reaches 30% of the thickness $R_2 - R_1$. The other parameters used for the calculations were $c = 0.15$, $\beta = 0.10$, $R_1 = 10$, $R_2 = 100$, and the same material constants as in Section 4.1 were considered. The equations corresponding to this problem, adapted from Ref. [21], are summarized in the Appendix.

It was first noted (Fig. 7) that without the localization limiter (that is for $\alpha = 0$), as the number of elements is increased, several points of localization develop, and these points

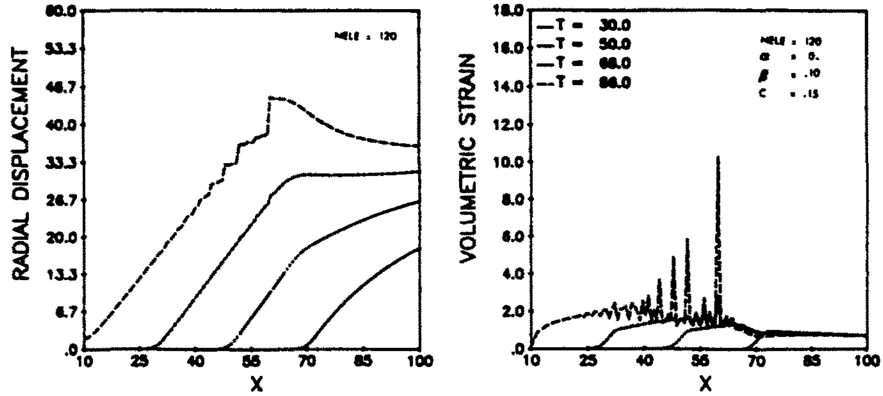

Fig. 7. Spherically symmetric problem, radial displacement and volumetric strain plots, without using the higher order term limiter ($\alpha = 0$).

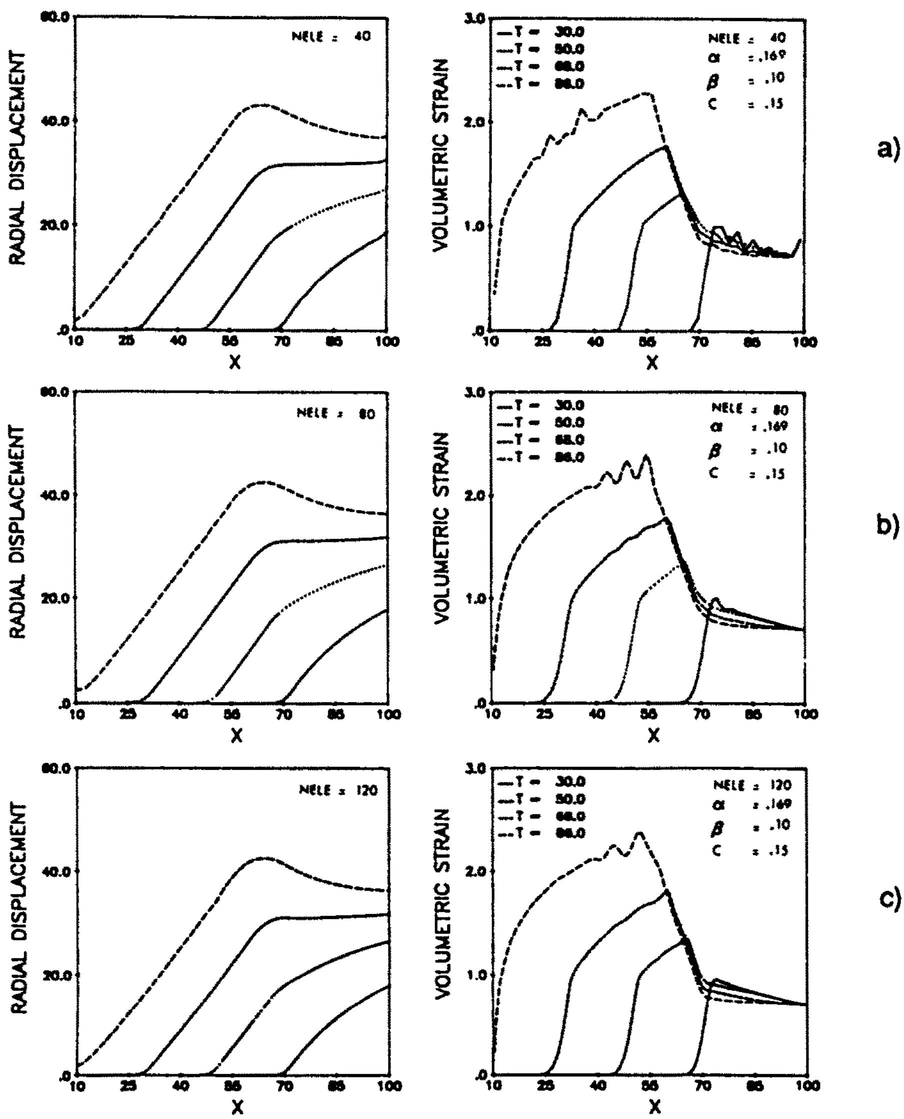

Fig. 8. Spherically symmetric problem, radial displacement and volumetric strain plots, with limiter ($\alpha = 0.169$): (a) 40 elements; (b) 80 elements; (c) 120 elements. For all plots, combination factor $c = 0.15$, damping coefficient $\beta = 0.15$.

change arbitrarily with mesh refinement, even in the presence of damping. These points of localization can be appreciated both in the volumetric strain plots, with the presence of spikes, and in the radial displacement plots, where sharp discontinuities indicate separation along a surface.

The next group of solutions (Figs 8(a)–(c)) examines the effect of the localization limiter. These solutions converge well with mesh refinement, and furthermore, they are very similar to those found with the imbricate element approach[21].

5. MATERIAL RATE DEPENDENCE SOLUTION TO THE SPHERICAL PROBLEM

It has been proposed recently by Needleman[7] that viscoplastic rate dependence eliminates the change of type associated with strain softening. His discussion, presented in the context of simple shearing, is equivalent to the tensile rod problem of Section 4.1. To summarize the formulation of Ref. [7], the incremental shear stress–shear strain relation is written as

$$\dot{\tau} = G(\dot{\gamma} - \dot{\gamma}^{p}). \tag{40}$$

Material rate dependence is introduced by writing the plastic strain rate $\dot{\gamma}^p$ in eqn (40) as

$$\dot{\gamma}^{p} = \dot{a} f(\tau, g) \tag{41}$$

where $\dot{a}$ is a material parameter and g a hardness parameter that evolves with accumulating plastic strain; f is given by a power law

$$f(\tau, g) = (\tau/g)^{1/m} \tag{42}$$

where m is the strain rate hardening exponent. In the numerical applications, g is taken to be a function of the effective plastic strain

$$\bar{\gamma} = \int |\dot{\gamma}^{p}| \, dt \tag{43}$$

here, a simple linear strain function is used

$$\dot{g} = H_1 \dot{\bar{\gamma}} \tag{44}$$

with $g(0) = \tau_0$.

The non-dimensional quantities τ_0/G, H_1/τ_0, τ/τ_0, $\gamma/(\tau_0/G)$ defined in Ref. [7] were also used here. Furthermore, we considered numerical values of the different parameters leading, in the rate-independent limit case, to a stress–strain curve similar to the curve considered here (Fig. 2(c)). A step-by-step integration of constitutive relations (40)–(44) reveals the role of the parameter m, as shown in Fig. 9: the rate-independent limit is approached as $m \to 0$, whereas, for increasing values of m, that is, for an increasingly viscous solid, the stress–strain curve "overshoots" the strain-softening branch, delaying the loss of positive definiteness of the stress–strain modulus.

The spherical problem of Section 4.2 was then solved introducing this material rate dependence, for different values of m. Parameters γ, τ and G are now interpreted as the volumetric strain, hydrostatic stress and bulk modulus, respectively. The shear modulus is considered as in Section 4.2 to be negligibly small (10^{-6}).

Figure 10 illustrates that, as m becomes smaller, the solution gets closer to the ones obtained with our localization limiter Fig. 8(b). However, the calculations become increasingly difficult: as pointed out in Ref. [7], when the constitutive equations are updated with a simple forward Euler method, the stable time step decreases and eventually vanishes with m. For example, the strain profile for $m = 0.005$ was obtained by running the problem with

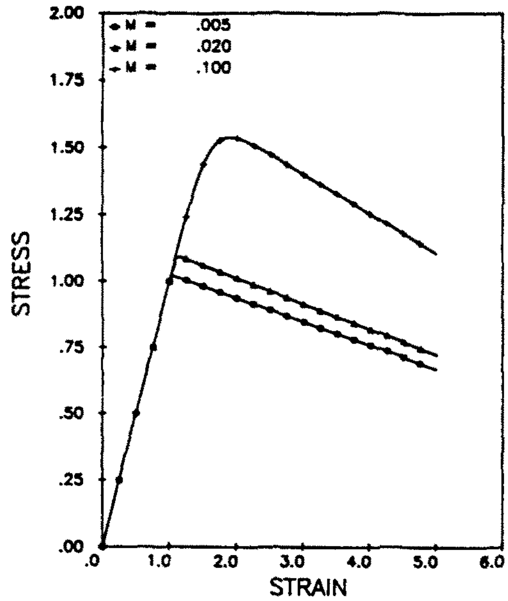

Fig. 9. Normalized stress (τ/τ_0)–normalized strain ($\gamma/(\tau_0/G)$) response for a rate-dependent material, for various values of the strain-hardening exponent m. $\tau_0/G = 0.008$, $H_1/\tau_0 = -10$, constant strain rate $\dot{\gamma}$ such that $\dot{\gamma}/\dot{a} = 10^2$.

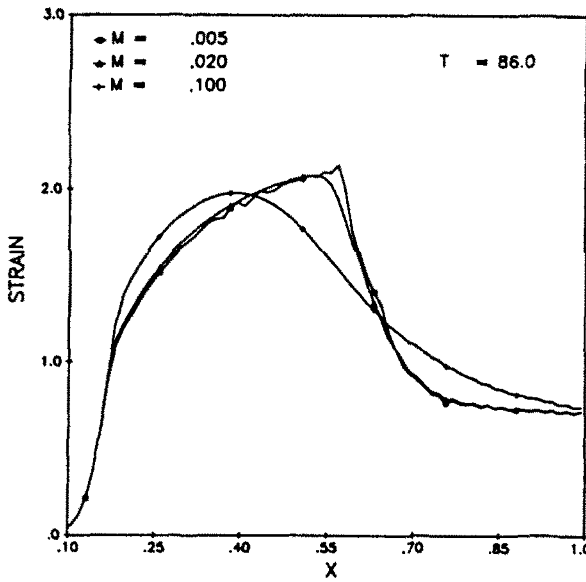

Fig. 10. Spherically symmetric problem, rate-dependent solid. Volumetric strains vs normalized radius x/R_2 at time $T = 86.0$, for various values of the strain-hardening exponent m.

a Courant number $CFL = 0.08$, but even this very small value leads to unstable calculations for $m = 0.001$.

6. CONCLUSIONS

The addition of a second derivative term to the strain expression leads to an efficient localization limiter for strain-softening materials, possessing the following desirable features: finite energy dissipation at arbitrary mesh refinement, development of a localized strain zone of finite size, independence of the calculations on mesh size. This theory may be considered nonlocal in that it emanates from a Taylor expansion of a non-local strain expression with a constant weight function.

It was shown by a Fourier analysis that this localization limiter yields a system of equations which attenuate short waves but cannot prevent the occurrence of instabilities in long waves. Since the initiation of strain softening appears to occur in very narrow zones, and the second derivatives are used only in the strain-softening (or localization) domains, this analysis implies that the growth of the strains in the localization domain will be bounded, rather than being unbounded as in the local, rate-independent medium. This was confirmed by numerical results.

A perturbation analysis reveals that the width of the zone in the strain-softening regime varies with the square root of α which determines the strength of the participation of the higher order derivative. This type of dependence is also indicated by a non-local interpretation of this limiter and was also verified numerically. This localization limiter was tested with success on one-dimensional rod problems, and on the more critical spherically symmetric problem, which emerges as a decisive test in evaluating the effectiveness of a localization limiter.

The localization limiting properties of the viscoplastic material model were also studied. It was shown that even in the spherically converging problem, this material model acts as a localization limiter. However, it is somewhat ineffective in explicit time integration problems since the stable time step becomes extremely small.

Several questions remain to be explored. A more thorough study of the additional boundary conditions introduced by the higher order terms is necessary. It would be desirable to relate the size of the localization zone (and therefore α) to the micromechanical behavior of the material, such as grain size, microcracking effects and other microstructural properties, and to give a physical interpretation of the additional terms.

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APPENDIX: GOVERNING EQUATIONS FOR THE SPHERICALLY SYMMETRIC PROBLEM

The strain–displacement relations are written as

$$\varepsilon_r = u_{,r}$$

$$\varepsilon_\theta = \varepsilon_z = \frac{u}{r}$$

$$\varepsilon_v = \varepsilon_r + 2\varepsilon_\theta$$

$$e_r = \tfrac{2}{3}(\varepsilon_r - \varepsilon_\theta)$$

$$e_\theta = \tfrac{1}{3}(\varepsilon_\theta - \varepsilon_r)$$

where u is the radial displacement (Fig. 3(b)), ε_r and ε_θ the radial and circumferential normal strains, ε_v the volumetric strain, and e_r and e_θ the deviatoric strains.

We define the modified quantities $\hat{\varepsilon}_r$, $\hat{\varepsilon}_\theta$, $\hat{\varepsilon}_v$, $\hat{e}_r$ and $\hat{e}_\theta$ in a way similar to eqn (9a). In particular, for the volumetric strain

$$\hat{\varepsilon}_v = \varepsilon_v + \alpha\varepsilon_{v,rr}.$$

The incremental stress–strain relations are given by

$$\dot{\sigma}_r = \hat{K}\dot{\hat{\varepsilon}}_v + 2\hat{G}\dot{\hat{e}}_r$$

$$\dot{\sigma}_\theta = \hat{K}\dot{\hat{\varepsilon}}_v + 2\hat{G}\dot{\hat{e}}_\theta$$

$$\dot{\tau}_r = K\dot{\varepsilon}_v + 2G\dot{e}_r$$

$$\dot{\tau}_\theta = K\dot{\varepsilon}_v + 2G\dot{e}_\theta.$$

The shear moduli $\hat{G}$ and G are assumed to be constant, while the tangent bulk moduli $\hat{K}$ and K and the stress–strain relations $(\sigma, \hat{\varepsilon}_v)$ and (τ, ε_v) are illustrated in Fig. 2. The total stress is given by

$$S_r = (1-c)s_r + c\tau_r$$

$$S_\theta = (1-c)\sigma_\theta + c\tau_\theta.$$

In the numerical applications, the shear moduli $\hat{G}$ and G are taken to be negligibly small (10^{-6}).